SCHWEIZER, B., and A. SKLAR
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Function Systems*

To KARL MENGER on the occasion of his 65th birthday

B. SCHWEIZER and A. SKLAR

Introduction

In a series of papers [9, 10, 11] which have appeared in this journal in the last few years, we have systematically developed the algebra of one-place functions[1]. One of our primary aims here was an axiomatic characterization of the ordered semigroup of functions whose domains and ranges are subsets of a fixed basic set. During the course of these investigations, two things became apparent: first, that it would be desirable to extend our studies to include the analogs of functions with general domains and ranges; second, that close connections exist between such a generalized algebra of functions and various other algebraic theories, notably the theory of inverse semigroups[2] and the theory of categories. The purpose of this paper is to make the extension and elucidate the connections.

The paper is divided into six sections. In the first, we present the axioms for our more general structures, which we call *function systems*, and develop some of their properties. In the second section we show that every function system can be represented as a system of one-place functions on a set, closed under composition, i.e., that every function system is embeddable in some primary function semigroup [11]. This theorem is a simultaneous generalization of our earlier representation theorem for function semigroups, the representation theorem of VAGNER and PRESTON for inverse semigroups, and the representation theorem of EILENBERG-MACLANE for categories. The remaining sections are devoted to a comparison of function systems with other algebraic structures. We show successively that the theory of function systems with an identity element coincides with our previous theory of function semigroups; that the theory of function systems in which every element has a subinverse coincides with the theory of inverse semigroups; and that the theory of function systems in which the inherent partial order is trivialized coincides with

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[1] The roots of these studies are to be found in various papers by K. MENGER, notably [6] and [7]. For a survey, summarizing 25 years of work by MENGER et al in the area of the algebra of functions, see [8].

[2] We are indebted to A. D. WALLACE for calling the possibility of this connection to our attention.

the theory of (small) categories. Finally, we indicate briefly how the theory of Brandt groupoids fits into our general framework.

Our usage of such terms as "functions", "mapping", "domain", "range", etc., generally conforms to that of P. M. COHN in [2]. Thus a *function* is a *consistent* set of ordered pairs, i.e., a set in which any two pairs with identical first elements have identical second elements and so coincide. The *domain*, $\mathrm{Dom}\,f$, of a function f is the set of first elements of all the pairs in f, and the *range* of f, $\mathrm{Ran}\,f$, is the set of all second elements[3]. For any x in $\mathrm{Dom}\,f$, the *value* $f(x)$ of f at x is the second element of the unique pair in f whose first element is x. A function f is a *restriction* of a function g (written $f \subsetneqq g$) if $\mathrm{Dom}\,f$ is a subset of $\mathrm{Dom}\,g$, and $f(x)=g(x)$ for all x in $\mathrm{Dom}\,f$. If f is a restriction of g, then g is an *extension* of f.

For any ordered pair (f,g) of functions, their *composite* $f\circ g$ is the (possibly empty) function defined by:

$$\mathrm{Dom}(f\circ g)=\{x\,|\,x \text{ in } \mathrm{Dom}\,g,\ g(x) \text{ in } \mathrm{Dom}\,f\},$$
$$(f\circ g)(x)=f(g(x)) \text{ for all } x \text{ in } \mathrm{Dom}(f\circ g).$$

Composition is the operation of forming the composite. For any set X, the identity function on X is denoted by j_X. Given a function f, we also write $\mathfrak{R}f$ for $j_{\mathrm{Dom}\,f}$ and $\mathfrak{L}f$ for $j_{\mathrm{Ran}\,f}$.

A *mapping* is a triple (A,B,f) in which A and B are sets, and f is a function whose domain is A and whose range is a subset of B. The *composite* $(A,B,f)\circ\circ(C,D,g)$ of two mappings (A,B,f) and (C,D,g) is defined if and only if $A=D$, in which case it is the mapping $(C,B,f\circ g)$.

1. Function systems

Definition 1. A *function system* (briefly, an *f-system*) is a quadruple $(A,\circ,L,R)$ consisting of a non-empty set A, a binary operation "$\circ$" and two unary operations L and R, all on A, subject to the following axioms:

Axiom 1. The pair $(A,\circ)$ is a semigroup, i.e., the binary operation "$\circ$" is associative.

Axiom 2. For all elements a of A, we have:

(a) $LRa=Ra,\ RLa=La$;

(b) $La\circ a=a=a\circ Ra$.

Axiom 3. For all pairs a, b of elements of A, we have:

(a) $L(a\circ b)=L(a\circ Lb),\ R(a\circ b)=R(Ra\circ b)$;

(b) $La\circ Rb=Rb\circ La$;

(c) $Ra\circ b=b\circ R(a\circ b)$.

These axioms are not completely independent. For example, the right half of A2b can be derived from the remaining identities via

$$a=La\circ a=RLa\circ a=a\circ R(La\circ a)=a\circ Ra.$$

However, this small degree of redundancy in the axioms will prove convenient later on (e.g., in §§ 3—5) and, as in the theory of function semigroups, also

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[3] The definitions could equally well be stated in terms of "classes" in the sense of VON NEUMANN-BERNAYS-GÖDEL set theory. In this paper, however, we shall confine ourselves to sets.

serves to bring out a far-reaching, though incomplete, symmetry inherent in every f-system. For, if in any expression involving elements of an f-system, the *dual* expression is defined to be the one obtained from the original by interchanging L and R and reversing the order of the factors in all $\circ$-products, then all the axioms, with the significant exception of A3c, are self-dual. The dual of A3c is the identity

$$\text{(D3c)} \qquad a \circ Lb = L(a \circ b) \circ a\,.$$

This identity is valid in certain f-systems (see §§ 4 and 5) but not in all[4].

Definition 2. An element s of an f-system $(A, \circ, L, R)$ is a *subidentity* if there is an element a in A such that $s = Ra$.

Theorem 1. *In a given f-system $(A, \circ, L, R)$, the following statements are equivalent:*

(A) *s is a subidentity,*
(B) $s = Ls$,
(C) *There is an element a in A such that $s = La$,*
(D) $s = Rs$.

Proof[5]. If s is a subidentity then $s = Ra$ for some a in A. Hence

$$Ls = LRa = Ra = s\,. \qquad \text{(A2a)}$$

Thus (A) implies (B); that (B) implies (C) is trivial; dually, (C) implies (D); and that (D) implies (A) is again trivial.

Corollary. For all a in A, $LLa = La$, $RRa = Ra$.

Theorem 2. *If s is a subidentity, then s is idempotent, i.e., $s \circ s = s$.*

Proof. If s is a subidentity then $s = Ls$. (T1)

Hence

$$s \circ s = Ls \circ s = s\,. \qquad \text{(A2b)}$$

With the aid of Theorem 1 we can restate A3b in the form of:

Theorem 3. *If s and t are subidentities, then s and t commute, i.e., $s \circ t = t \circ s$.*

It should be noted that f-systems, like function semigroups, can contain idempotent elements that are not subidentities, e.g., any element $a (\neq La)$ such that $a \circ La = La$; such idempotents will in general commute neither with subidentities nor with each other.

Theorem 4. *If s and t are subidentities then $s \circ t$ is a subidentity.*

Proof. We have

$$\begin{aligned} s \circ t &= Rs \circ t = t \circ R(s \circ t) && \text{(T1, A3c)} \\ &= Rt \circ R(s \circ t) = R(s \circ t) \circ R(t \circ R(s \circ t)) && \text{(T1, A3c)} \\ &= R(s \circ t) \circ R(s \circ t) = R(s \circ t)\,. && \text{(T1, T2)} \end{aligned}$$

Corollary 1. The set of all subidentities in A is closed under the operations "$\circ$", L, and R.

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[4] The system of all binary relations on a set, with composition (or relative product) and $\mathfrak{R}$ and $\mathfrak{L}$ as in the Introduction, does *not* form an f-system. In fact, binary relations satisfy all the axioms with the significant exception of A3c.

[5] Here and in subsequent proofs, the marks within parentheses on the right refer to the axioms, theorems, etc., used to justify the various steps.

Corollary 2. For any a, b in A we have

(i) $L(Ra \circ Rb) = R(Ra \circ Rb) = Ra \circ Rb$,

(ii) $L(La \circ Lb) = R(La \circ Lb) = La \circ Lb$,

(iii) $L(Ra \circ Lb) = R(Ra \circ Lb) = Ra \circ Lb$.

Implicit in the axioms for an f-system is an order relation which is made explicit via the following:

Definition 3. If a, b are elements of an f-system then $a \subseteqq b$ if and only if $a = b \circ Ra$.

Theorem 5. *The relation "$\subseteqq$" is a (non-strict) partial order.*

Proof. (i) From $a = a \circ Ra$, it follows that $a \subseteqq a$. (A2b)

(ii) Let $a \subseteqq b$ and $b \subseteqq a$. Then we have

$$\begin{aligned} a &= b \circ Ra = (b \circ Rb) \circ Ra && \text{(D3, A2b)} \\ &= b \circ (Rb \circ Ra) = b \circ (Ra \circ Rb) && \text{(A1, T1, T3)} \\ &= (b \circ Ra) \circ Rb = a \circ Rb = b\,. && \text{(A1, D3)} \end{aligned}$$

Note that this is the first time we use the associativity of "$\circ$".

(iii) Let $a \subseteqq b$ and $b \subseteqq c$. Then we have:

$$\begin{aligned} a &= b \circ Ra = c \circ Rb \circ Ra && \text{(D3)} \\ &= c \circ R(Rb \circ Ra) = c \circ R(b \circ Ra) = c \circ Ra\,, && \text{(C4.2, A3a, D3)} \end{aligned}$$

whence $a \subseteqq c$.

Theorem 6. *For all a, b in A, we have $L(a \circ b) \subseteqq La$ and $R(a \circ b) \subseteqq Rb$.*

Proof. (i) $L(a \circ b) = L(La \circ a \circ b) = L(La \circ L(a \circ b))$ (A2b, A3a)

$= La \circ L(a \circ b) = La \circ RL(a \circ b)\,,$ (C4.2, A2a)

whence $L(a \circ b) \subseteqq La$. (D3)

(ii) $R(a \circ b) = R(a \circ b \circ Rb) = R(R(a \circ b) \circ Rb)$ (A2b, A3a)

$= R(a \circ b) \circ Rb = Rb \circ R(a \circ b)$ (C4.2, T1, T3)

$= Rb \circ RR(a \circ b)\,,$ (C1)

whence $R(a \circ b) \subseteqq Rb$. (D3)

Corollary. If $a \subseteqq b$, then $La \subseteqq Lb$ and $Ra \subseteqq Rb$.

Proof. Since $a = b \circ Ra$, we have $La = L(b \circ Ra) \subseteqq Lb$, (D3, T6)

and

$$\begin{aligned} Ra &= R(b \circ Ra) = R(Rb \circ Ra) && \text{(A3a)} \\ &= R(Ra \circ Rb) \subseteqq RRb = Rb\,. && \text{(T3, T6, C1)} \end{aligned}$$

Theorem 7. *If s is a subidentity and $Ra \subseteqq s$, then $a \circ s = a$. If s is a subidentity and $La \subseteqq s$, then $s \circ a = a$.*

Proof. If s is a subidentity and $Ra \subseteqq s$, then

$$\begin{aligned} a \circ s &= a \circ Ra \circ s = a \circ s \circ Ra = a \circ s \circ RRa && \text{(A2b, T1, T3, C1)} \\ &= a \circ Ra = a\,. && \text{(D3, A2b)} \end{aligned}$$

If s is a subidentity and $La \subseteqq s$, then

$$s \circ a = s \circ La \circ a = s \circ RLa \circ a = La \circ a = a\,.$$

Corollary. If $Ra \subseteq Lb$ then $L(a \circ b) = La$. If $Lb \subseteq Ra$ then $R(a \circ b) = Rb$.

The second half of the above corollary (but generally not the first half) has a valid converse, namely:

Theorem 8. *If* $R(a \circ b) = Rb$, *then* $Lb \subseteq Ra$.

Proof. $Lb = L(b \circ Rb) = L(b \circ R(a \circ b)) = L(Ra \circ b)$ (A2b, A3c)

$= L(Ra \circ Lb) = Ra \circ Lb = Ra \circ RLb.$ (A3a, C4.2, A2a)

Theorem 9. *If s is a subidentity and $a \subseteq s$, then a is a subidentity and* $a \circ s = s \circ a = a$.

Proof. We have $a = s \circ Ra$. Thus, a, being a product of subidentities, is itself a subidentity. The rest follows from T1 and T7.

A semigroup $(A, \circ)$ can be an f-system under two distinct pairs of unary operations (L_1, R_1) and (L_2, R_2). If this is the case then, as the following theorem shows, the induced partial orders must differ.

Theorem 10. *Let $(A, \circ, L_1, R_1)$ and $(A, \circ, L_2, R_2)$ be two f-systems (over the same semigroup $(A, \circ)$) such that the partial orders induced in each of them via* D3 *coincide, i.e., such that for all a, b in A the statements: "$a \subseteq b$", "$a = b \circ R_1 a$", "$a = b \circ R_2 a$" are equivalent. Then $R_1 = R_2$ and $L_1 = L_2$.*

Proof. By hypothesis, for any a in A we have

(†) $$a = a \circ R_1 a = a \circ R_2 a\,.$$

In particular, $R_1 a = R_1 a \circ R_2 R_1 a$ and $R_2 a = R_2 a \circ R_1 R_2 a$. Next, for any a in A,

$$R_1 a \circ R_2 a = R_2 a \circ R_1(a \circ R_2 a) = R_2 a \circ R_1 a\,. \qquad \text{(A3c)}$$

Hence

$$R_1 a = R_1 a \circ R_2 R_1 a = R_1 R_1 a \circ R_2 R_1 a = R_2 R_1 a \circ R_1 R_1 a \qquad \text{(C1)}$$

which implies $R_1 a \subseteq R_2 R_1 a$, so that

$$R_1 a = R_2 R_1 a \circ R_2 R_1 a = R_2 R_1 a\,; \qquad \text{(T1, T2)}$$

similarly, $R_2 a = R_1 R_2 a$. But in view of (†),

$$R_1 a = R_1(a \circ R_2 a) \subseteq R_1 R_2 a = R_2 a\,. \qquad \text{(T6)}$$

Similarly, $R_2 a \subseteq R_1 a$, whence $R_1 a = R_2 a$.

Using this result, we obtain

$$L_1 L_2 a = L_1 R_2 L_2 a = L_1 R_1 L_2 a = R_1 L_2 a = R_2 L_2 a = L_2 a\,. \qquad \text{(A2a)}$$

But

$$L_1 a = L_1(L_2 a \circ a) \subseteq L_1 L_2 a = L_2 a\,. \qquad \text{(A3a, T6)}$$

Similarly, $L_2 a \subseteq L_1 a$, whence $L_1 a = L_2 a$ and the proof is complete.

Corollary. If $(A, \circ, L_1, R)$ and $(A, \circ, L_2, R)$ are f-systems, then $L_1 = L_2$.

Definition 4. An element n of a semigroup $(A, \circ)$ is an *identity* if $n \circ a = a \circ n = a$ for all a in A. An element ϕ of $(A, \circ)$ is a *left-zero* if $\phi \circ a = \phi$ for all a in A, a *right-zero* if $a \circ \phi = \phi$ for all a in A and a *zero* if it is both a left- and a right-zero.

These notions are standard in the theory of semigroups. Furthermore, it is well-known [1; p. 31] that identities and zeros, whenever they exist, are unique.

Theorem 11. *An element n of an f-system $(A, \circ, L, R)$ is an identity if and only if $n \circ Ra = Ra$ for all a in A, i.e., if and only if all subidentities s of the f-system satisfy the condition $s \subseteq n$.*

Proof. If n is an identity and s is a subidentity then, trivially, $s = n \circ s = n \circ Rs$, whence $s \subseteq n$. Conversely, if $s \subseteq n$, or equivalently, if $n \circ s = s$ for all subidentities s in A, then since $Rn \subseteq n$ we have $Rn = n \circ RRn = n \circ Rn = n$, whence n is a subidentity. Next we have

$$a \circ n = a \circ Ra \circ n = a \circ n \circ Ra = a \circ Ra = a, \qquad \text{(A2b, T1, T3)}$$

for all a in A; and finally,

$$n \circ a = n \circ La \circ a = La \circ a = a, \qquad \text{(A2b)}$$

for all a in A, and the proof is complete.

Combining Theorems 9 and 11 yields the following:

Corollary. If an f-system has an identity n and if $n_1 \subseteq n$ then n_1 is a subidentity.

The preceding theorem and corollary together justify the use of the term "subidentity" for elements of the form Ra or La. It should be emphasized, however, that an f-system (which always has subidentities) need not have an identity. In the other direction, any semigroup $(A, \circ)$ with an identity n can immediately be converted into an f-system by defining $La = Ra = n$ for all a in A.

Definition 5. An element ϕ of an f-system $(A, \circ, L, R)$ is a *null element* if it is both a left-zero ($\phi \circ a = \phi$ for all a) and a subidentity ($\phi = R\phi$).

From the remark before Definition 5 it follows that an f-system can have a zero which may not be a null element: e.g., if $(A, \circ)$ is a semigroup with zero z and identity n, we may have $Lz = Rz = n \neq z$. In the other direction we have:

Theorem 12. *If an f-system $(A, \circ, L, R)$ has a null element ϕ, then ϕ is a zero of the f-system. Furthermore, ϕ is the minimal element of the f-system, i.e., $\phi \subseteq a$ for all a in A.*

Proof. For all a in A, we have

$$a \circ \phi = a \circ R\phi = a \circ R(\phi \circ a) = R\phi \circ a = \phi \circ a = \phi. \qquad \text{(D5, A3c)}$$

Hence ϕ is a right-zero, and consequently a zero. It follows that for any a in A, $a \circ R\phi = a \circ \phi = \phi$, whence $\phi \subseteq a$ and ϕ is minimal.

Corollary 1. The statements "$a = \phi$", "$Ra = \phi$", "$La = \phi$" are equivalent.

Corollary 2. $a \circ b = \phi$ if and only if $a = \phi$ or $b = \phi$ or $Ra \circ Lb = \phi$.

The proof of Theorem 12 shows that an element which is a subidentity and a right-zero is minimal. However, the example in § 6 of [9] shows that there exist f-systems which contain elements that are subidentities and right-zeros, but not left-zeros, so that a minimal element need not be a null element.

Definition 6. Let a be an element of an f-system. An element b of the f-system is a *left-subinverse* of a (briefly, $b[LS]a$) if $b \circ a = Ra$ and $Rb \subseteq La$; b is a *right-subinverse* of a (briefly, $b[RS]a$) if $a \circ b = La$ and $Lb \subseteq Ra$; and b is a *subinverse* of a (briefly, $b[S]a$) if both $b[LS]a$ and $b[RS]a$.

Theorem 13. *If $b[LS]a$ then $Lb = Ra$. If $b[RS]a$ then $Rb = La$.*

Proof. If $b[LS]a$, then since $Rb \subseteq La$, we have $b = b \circ La$. (T7)

Hence

$$Lb = L(b \circ La) = L(b \circ a) = LRa = Ra\,. \tag{A3a}$$

Dually, $b[RS]a$ yields $Rb = La$.

Corollary 1. If $b[S]a$, then $Lb = Ra$, $Rb = La$, and $a[S]b$.

Corollary 2. If $b[S]a$ and $c[S]a$, then $b = c$. Thus, subinverses, whenever they exist, are unique.

Proof. We have $b = b \circ Rb = b \circ La = b \circ a \circ c = Ra \circ c = Lc \circ c = c$.

Definition 7. An f-system has the *left-subinverse property* (briefly, the *LS-property*) if each of its elements has a left-subinverse, has the *right-subinverse property* (briefly, the *RS-property*) if each of its elements has a right-subinverse, and has the *subinverse property* if each of its elements has a (necessarily unique) subinverse.

The interrelationship of these properties will be discussed later, particularly in § 4.

2. Functions and the representation theorem

If a set $\mathscr{S}$ of functions is closed under $\circ$, $\mathfrak{L}$, and $\mathfrak{R}$ (see the Introduction), then it is a straightforward matter to verify that $(\mathscr{S}, \circ, \mathfrak{L}, \mathfrak{R})$ is an f-system. In particular, if A is a non-empty set and $\mathscr{I}(A)$ is the set of all functions f such that both $\operatorname{Dom} f$ and $\operatorname{Ran} f$ are subsets of A, then $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$ is an f-system, the *primary function semigroup* (on A) as defined in [11]. There we discussed primary function semigroups in detail and characterized them abstractly. Here we need only recall that any primary function semigroup has an identity (the identity function on A), a null element (the empty function on A), and has the RS-, but generally not the LS-property.

A subset $\mathscr{S}$ of $\mathscr{I}(A)$ may be closed under composition, $\mathfrak{L}$, and $\mathfrak{R}$; in this case the quadruple $(\mathscr{S}, \circ, \mathfrak{L}, \mathfrak{R})$ is a *sub-f-system of* $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$ *in the strict sense*. If $\mathscr{S}$ is closed under composition, under $\mathfrak{R}$, and under a unary operation $\mathfrak{L}_1$, which may differ from $\mathfrak{L}$, and if the quadruple $(\mathscr{S}, \circ, \mathfrak{L}_1, \mathfrak{R})$ is an f-system, then we say that this f-system is a *sub-f-system of* $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$ *in the wide sense*. In this case, the corollary to Theorem 10 shows that $\mathfrak{L}_1$ can actually differ from $\mathfrak{L}$ if and only if $\mathscr{S}$ is *not* closed under $\mathfrak{L}$.

For our representation theorem (Theorem 14) we require two lemmas. These, as well as Theorem 14 itself, are counterparts of Lemmas 1, 2, 3 and Theorem 48 of [10].

Lemma 1. For any three elements a, b, c of an f-system, we have $Lc \subseteq R(a \circ b)$ if and only if $Lc \subseteq Rb$ and $L(b \circ c) \subseteq Ra$.

Proof. (1) Suppose $Lc \subseteq R(a \circ b)$. Then, since by T6, $R(a \circ b) \subseteq Rb$, we have $Lc \subseteq Rb$. Next, since $Lc = R(a \circ b) \circ Lc$, we have:

$$\begin{aligned} L(b \circ c) = L(b \circ Lc) = L(b \circ R(a \circ b) \circ Lc) &\subseteq L(b \circ R(a \circ b)) && \text{(A3a, T6)} \\ = L(Ra \circ b) \subseteq LRa = Ra\,. & && \text{(A3c, T6, A2a)} \end{aligned}$$

(2) Now suppose $Lc \subseteq Rb$ and $L(b \circ c) \subseteq Ra$. Then, first of all,

$$L(b \circ Lc) = L(b \circ c) \subseteq Ra\,, \tag{A3a}$$

whence

$$b \circ Lc = Ra \circ b \circ Lc . \qquad \text{(T7)}$$

Thus

$$\begin{aligned} Lc &= Rb \circ Lc = R(Rb \circ Lc) = R(b \circ Lc) = R(Ra \circ b \circ Lc) && \text{(T7, C4.2, A3a)} \\ &= R(a \circ b \circ Lc) = R(R(a \circ b) \circ Lc) = R(a \circ b) \circ Lc , && \text{(A3a, C4.2)} \end{aligned}$$

whence $Lc \subsetneq R(a \circ b)$.

Lemma 2. If a and b are distinct elements of an f-system, then either $a \neq b \circ Ra$ or $b \neq a \circ Rb$.

Proof. If $a = b \circ Ra$ and $b = a \circ Rb$, then $a \subsetneq b$ and $b \subsetneq a$, whence $a = b$ by Theorem 5, contradicting the hypothesis of the lemma.

Theorem 14. *For any element a of an f-system $(A, \circ, L, R)$, let f_a denote the function defined by:*

$$\operatorname{Dom} f_a = \{c \mid c \in A,\ c \neq \phi,\ Lc \subsetneq Ra\} ,$$
$$f_a(c) = a \circ c \quad \textit{for all} \quad c \in \operatorname{Dom} f_a ,$$

where "$\subsetneq$" denotes the partial order introduced in Definition 3 and ϕ is the null element, if any, of A. Let $\mathscr{I}_A$ denote the set of all such functions f_a. In $\mathscr{I}_A$, let "$\circ$" denote composition and "$\subsetneq$" restriction, as in the Introduction. Finally, let unary operations L and R be defined in $\mathscr{I}_A$ by: $Lf_a = f_{La}$, $Rf_a = f_{Ra}$ for all a in A. Then $(\mathscr{I}_A, \circ, L, R)$ is an f-system order-isomorphic to the f-system $(A, \circ, L, R)$ under the correspondence $a \leftrightarrow f_a$.

Proof. (1) The correspondence $a \to f_a$ is a homomorphism. For any two elements a, b in A, we have

$$\begin{aligned} \operatorname{Dom}(f_a \circ f_b) &= \{c \mid c \in \operatorname{Dom} f_b, f_b(c) \in \operatorname{Dom} f_a\} \\ &= \{c \mid c \in A, c \neq \phi, Lc \subsetneq Rb, b \circ c \in \operatorname{Dom} f_a\} \\ &= \{c \mid c \in A, c \neq \phi, Lc \subsetneq Rb, L(b \circ c) \subsetneq Ra\} . \end{aligned}$$

But by Lemma 1, this last set is the same as the set

$$\{c \mid c \in A, c \neq \phi, Lc \subsetneq R(a \circ b)\} = \operatorname{Dom} f_{a \circ b} ,$$

whence $\operatorname{Dom}(f_a \circ f_b) = \operatorname{Dom} f_{a \circ b}$. Next, for any element c belonging to $\operatorname{Dom} f_{a \circ b} = \operatorname{Dom}(f_a \circ f_b)$ we have

$$\begin{aligned} f_{a \circ b}(c) &= (a \circ b) \circ c = a \circ (b \circ c) = f_a(b \circ c) \\ &= f_a(f_b(c)) = (f_a \circ f_b)(c) . \end{aligned}$$

Thus $f_{a \circ b} = f_a \circ f_b$ and the correspondence $a \to f_a$ is a homomorphism.

(2) The correspondence is order-preserving in both directions. Let $a \subsetneq b$ in A. Then $Ra \subsetneq Rb$, by Corollary 6, so if $Lc \subsetneq Ra$, then $Lc \subsetneq Rb$. But this implies that if $c \in \operatorname{Dom} f_a$, then $c \in \operatorname{Dom} f_b$. Thus for any $c \in \operatorname{Dom} f_a$, we have

$$f_a(c) = a \circ c = b \circ Ra \circ c = b \circ c = f_b(c) , \qquad \text{(D3, T7)}$$

whence $f_a \subsetneq f_b$.

Next, if $a = \phi$, then f_a is empty; and if $a \neq \phi$, then $Ra \in \operatorname{Dom} f_a$, whence f_a is not empty. Thus f_a is empty if and only if $a = \phi$.

Now, turning to the converse, suppose $f_a \subseteqq f_b$. If f_a is empty, then $a = \phi$, whence $a \subseteqq b$ by Theorem 12. If f_a is not empty, then $Ra \in \operatorname{Dom} f_a$, whence $Ra \in \operatorname{Dom} f_b$ and $f_a(Ra) = f_b(Ra)$. But $f_a(Ra) = a \circ Ra = a$ and $f_b(Ra) = b \circ Ra$. Thus $a = b \circ Ra$ and $a \subseteqq b$.

(3) The correspondence is one-to-one. To prove this, we need only show that $a \neq b$ implies $f_a \neq f_b$. This is immediate if either $a = \phi$ or $b = \phi$. If $a \neq \phi$ and $b \neq \phi$ then $Ra \in \operatorname{Dom} f_a$ and $Rb \in \operatorname{Dom} f_b$. If now $Ra \notin \operatorname{Dom} f_b$ or $Rb \notin \operatorname{Dom} f_a$ then surely $f_a \neq f_b$. There remains only the case in which Ra and Rb both belong to $\operatorname{Dom} f_a \cap \operatorname{Dom} f_b$. But then it follows from Lemma 2 that either $f_a(Ra) \neq f_b(Ra)$ or $f_a(Rb) \neq f_b(Rb)$; and again $f_a \neq f_b$. This completes the proof.

Since $\mathscr{I}_A$ is a subset of $\mathscr{I}(A)$, it is natural to ask whether $(\mathscr{I}_A, \circ, L, R)$ is a sub-f-system of $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$. The answer is contained in the following theorem, which we state without proof (for details, see Theorem 61 of [11]):

Theorem 15. *The f-system $(\mathscr{I}_A, \circ, L, R)$ of Theorem 14 is always a sub-f-system of $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$ in the wide sense, i.e., R and $\mathfrak{R}$ coincide. A necessary and sufficient condition that $(\mathscr{I}_A, \circ, L, R)$ be a sub-f-system of $(\mathscr{I}(A), \circ, \mathfrak{L}, \mathfrak{R})$ in the strict sense is that $(\mathscr{I}_A, \circ, L, R)$, or equivalently $(A, \circ, L, R)$, have the RS-property.*

3. Function semigroups

To compare f-systems with function semigroups, we begin with:

Lemma 3. If an f-system $(A, \circ, L, R)$ has an identity n, then for all a in A and all $s \subseteqq n$ we have $a \circ s \subseteqq a$ and $s \circ a \subseteqq a$.

Proof. We have, first of all

$$a \circ s = a \circ Ra \circ s = a \circ R(Ra \circ s) = a \circ R(a \circ s) \,, \quad \text{(A2b, C4.2, A3a)}$$

whence

$$a \circ s \subseteqq a \,.$$

Next,

$$s \circ a = Rs \circ a = a \circ R(s \circ a), \quad \text{whence} \quad s \circ a \subseteqq a \,. \qquad \text{(T1, A3c)}$$

Theorem 16. *An f-system with an identity n is a function semigroup, and conversely.*

Proof. An f-system $(A, \circ, L, R)$ with an identity n has the following properties:

(1) A is partially ordered by "$\subseteqq$". (T5)

(2) $(A, \circ)$ is a semigroup with unit n. (A1)

(3) If $a \subseteqq b$ then there exists $n_1 \subseteqq n$ such that $a = b \circ n_1$. (D3, T11)

(4) If $n_1 \subseteqq n$ then $a \circ n_1 \subseteqq a$ and $n_1 \circ a \subseteqq a$. (L3)

(5) $La \circ a = a = a \circ Ra$. (A2b)

(6) $L(a \circ b) \subseteqq La$, $R(a \circ b) \subseteqq Rb$. (T6)

(7) If $a \subseteqq n$, then $La \subseteqq a$ and $Ra \subseteqq a$. (C11, T1)

(8) $L(a \circ b) = L(a \circ Lb)$, $R(a \circ b) = R(Ra \circ b)$. (A3a)

But statements (1) through (8) constitute the definition of a function semigroup ([10], Definition 12).

Conversely, the various parts of A1 and A2 are axioms or theorems in the theory of function semigroups as developed in [9] and [10], the correspondence being as follows: A1 with Axiom 2; A2a with Corollary 3; A2b with Axiom 4a; A3a with Axiom 6; A3b with Theorem 2 and Theorem 5; and A3c with Theorem 43.

4. Inverse semigroups

If an element of an f-system has a subinverse then Corollary 2 of Theorem 13 shows that this subinverse is unique. We will call an element that has a subinverse *invertible* and denote the unique subinverse of the invertible element a by Sa. The corollaries to Theorem 13 then take the form: $SSa=a$, $a\circ Sa=La=RSa$, $Sa\circ a=Ra=LSa$. Note also that if s is a subidentity, then s is invertible and $Ss=s$.

Theorem 17. *If a and b are invertible, then $a\circ b$ is invertible and $S(a\circ b)=Sb\circ Sa$.*

Proof. Since $R(a\circ b)\subseteqq Rb$, we have $R(a\circ b)=Rb\circ R(a\circ b)$. Thus

$$\begin{aligned} Sb\circ Sa\circ a\circ b&=Sb\circ Ra\circ b=Sb\circ b\circ R(a\circ b)\\ &=Rb\circ R(a\circ b)=R(a\circ b)\,. \end{aligned}$$

Next,

$$\begin{aligned} R(a\circ b)&=LR(a\circ b)=L(Sb\circ Ra\circ b)=L(Sb\circ Ra\circ Lb)=L(Sb\circ Lb\circ Ra)\\ &=L(Sb\circ RSb\circ Ra)=L(Sb\circ LSa)=L(Sb\circ Sa)\,. \end{aligned}$$

Dually, we obtain $a\circ b\circ Sb\circ Sa=R(Sb\circ Sa)=L(a\circ b)$, from which the desired conclusion follows.

Theorem 18. *If a is an invertible element of an f-system, then $a\circ Lb=L(a\circ b)\circ a$ for all elements b of the f-system.*

Proof. Both a and Lb are invertible. Hence $a\circ Lb$ is invertible and

$$S(a\circ Lb)=SLb\circ Sa=Lb\circ Sa\,.$$

Consequently,

$$\begin{aligned} L(a\circ b)\circ a&=L(a\circ Lb)\circ a=RS(a\circ Lb)\circ a\\ &=R(Lb\circ Sa)\circ a=a\circ R(Lb\circ Sa\circ a)\\ &=a\circ R(Lb\circ Ra)=a\circ Lb\circ Ra=a\circ Ra\circ Lb\\ &=a\circ Lb\,. \end{aligned}$$

Corollary. If an f-system $(A,\circ,L,R)$ has the subinverse property, then the identity D3c is valid for all pairs of elements of A.

Theorem 19. *If an f-system $(A,\circ,L,R)$ has the RS-property and if D3c is valid for all pairs of elements of A, then the f-system has the subinverse property.*

Proof. Let a be any element of A and let $b[RS]a$. Then

$$a\circ b=La=Rb \quad\text{and}\quad R(a\circ Lb)\subseteqq RLb=Lb\,.$$

Hence $a \circ Lb[LS]b$. Now b has a right-subinverse, say c. By definition $Lc \subseteqq Rb = La$ and $b \circ c = Lb$, whence

$$c = La \circ c = a \circ b \circ c = a \circ Lb\,,$$

i.e., $a \circ Lb[RS]b$. Therefore $a \circ Lb[S]b$, whence $a \circ Lb$ is invertible with $S(a \circ Lb) = b$. But by D3c,

$$a \circ Lb = L(a \circ b) \circ a = La \circ a = a\,,$$

and a is invertible.

The following result, which is essentially the dual of the preceding theorem, is due to S. Penner:

Theorem 20. *If an f-system $(A, \circ, L, R)$ has the LS-property, then the f-system has the subinverse property.*

Proof. Let a be any element of A, and let $b[LS]a$. By arguments dual to those in the proof of T19, it follows that $Rb \circ a$ is invertible with $S(Rb \circ a) = b$. But (now without use of D3c)

$$Rb \circ a = a \circ R(b \circ a) = a \circ RRa = a \circ Ra = a\,,$$

whence a is invertible, and the theorem is proved.

Since the subinverse property implies both the RS- and the LS-property, the results of Theorems 18—20 can be combined as follows:

Theorem 21. *The following statements are equivalent:*

(i) The f-system $(A, \circ, L, R)$ has the RS-property and the identity D3c is valid throughout the f-system.

(ii) The f-system $(A, \circ, L, R)$ has the LS-property.

(iii) The f-system $(A, \circ, L, R)$ has the subinverse property.

We now turn to inverse semigroups. We shall follow Clifford and Preston [1, pp. 26—33]. Thus an inverse semigroup can be defined as a triple $(A, \circ, ^{-1})$, where $(A, \circ)$ is a semigroup and $^{-1}$ is the operation fixed by the requirement that for any a in A, there is a unique a^{-1} in A such that $a \circ a^{-1} \circ a = a$ and $a^{-1} \circ a \circ a^{-1} = a^{-1}$.

Theorem 22. *Let $(A, \circ, L, R)$ be an f-system with the subinverse property, and let $^{-1}$ be the operation defined by: $a^{-1} = Sa$. Then $(A, \circ, ^{-1})$ is an inverse semigroup.*

Proof. For any a in A we have

$$a \circ Sa \circ a = a \circ Ra = a, \quad Sa \circ a \circ Sa = Sa \circ RSa = Sa\,,$$

whence a and Sa are relative inverses. Thus we need only show that Sa is the unique relative inverse of a. To this end, let b be any relative inverse of a, i.e., suppose that

$$a \circ b \circ a = a, \quad b \circ a \circ b = b\,.$$

Then we have

$$\begin{aligned} a \circ b &= La \circ a \circ b = L(a \circ b \circ a) \circ a \circ b \\ &= a \circ L(b \circ a) \circ b = a \circ b \circ La \\ &= a \circ b \circ a \circ Sa = a \circ Sa = La\,. \end{aligned}$$

Dually, we have $b \circ a = Ra$. And similarly, we obtain $a \circ b = Rb$, $b \circ a = Lb$. Hence $b = Sa = a^{-1}$ and the proof is complete.

In the other direction we have:

Theorem 23. *Let* $(A, \circ, {}^{-1})$ *be an inverse semigroup. Define operations* L *and* R *by:* $La = a \circ a^{-1}$, $Ra = a^{-1} \circ a$, *for every* a *in* A. *Then* $(A, \circ, L, R)$ *is an* f*-system with the subinverse property.*

Proof. We first note that A2b simply reduces to the definition of a^{-1}. Next, since La and Ra are idempotent [1; p. 26] and since any idempotent element is its own relative inverse, we have $(La)^{-1} = La$ and $(Ra)^{-1} = Ra$. Thus

$$LRa = Ra \circ (Ra)^{-1} = Ra \circ Ra = Ra\,,$$

and

$$RLa = (La)^{-1} \circ La = La \circ La = La\,.$$

This verifies A2a. As for A3, we first note that $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ [1; L1.18, p. 30]. Therefore

$$\begin{aligned} L(a \circ b) &= (a \circ b) \circ (a \circ b)^{-1} = a \circ b \circ b^{-1} \circ a^{-1} \\ &= a \circ Lb \circ a^{-1} = a \circ Lb \circ Lb \circ a^{-1} \\ &= a \circ Lb \circ (Lb)^{-1} \circ a^{-1} = a \circ Lb \circ (a \circ Lb)^{-1} \\ &= L(a \circ Lb)\,. \end{aligned}$$

Dually, $R(a \circ b) = R(Ra \circ b)$. This proves A3a. The validity of A3b is a consequence of T1.17i in [1; p. 28]. Finally using the already established parts of A2 and A3, we have

$$\begin{aligned} Ra \circ b &= Ra \circ Lb \circ b = Lb \circ Ra \circ b = b \circ b^{-1} \circ a^{-1} \circ a \circ b \\ &= b \circ (a \circ b)^{-1} \circ (a \circ b) = b \circ R(a \circ b)\,. \end{aligned}$$

Thus all of A2 and A3 are valid and $(A, \circ, L, R)$ is an f-system which obviously has the subinverse property.

Combining Theorems 22 and 23 yields the result that inverse semigroups coincide with f-systems with the subinverse property. Since D3c is valid in any inverse semigroup, the entire theory is self-dual. We also note that T14, when specialized to inverse semigroups reduces to the (dual of the) representation theorem of VAGNER and PRESTON as given in [1, pp. 30—31].

5. Categories

To compare f-systems with categories we begin by introducing the notion of a categorical semigroup.

Definition 8. *A categorical semigroup* is a quadruple $(A, \circ, L, R)$ satisfying Axioms 1 and 2, possessing a zero element ϕ such that $R\phi = \phi$, and subject to the additional condition:

(C) For all a, b in A, $a \circ b \neq \phi$ if and only if $a \neq \phi$, $b \neq \phi$ and $Ra = Lb$.

Theorem 24. *A categorical semigroup is an* f*-system; conversely, any* f*-system which has a zero element* ϕ *and in which Condition C is valid is a categorical semigroup.*

Proof. We first note that T1 and T2 are independent of A3, therefore valid in any categorical semigroup; and second, that if one of the three elements, a, Ra, La is equal to ϕ, then so are the other two. Now let a, b be any pair of elements belonging to a categorical semigroup. To verify the various parts of A3, we have:

(A3a) If $a \circ b = \phi$, then either $a = \phi$ or $b = \phi$ or $Ra \neq Lb$. In each case we have at once that

$$L(a \circ b) = L(a \circ Lb) = R(a \circ b) = R(Ra \circ b) = \phi\,.$$

If $a \circ b \neq \phi$, then $Ra = Lb$, whence $L(a \circ Lb) = L(a \circ Ra) = La$. Next, $La \circ a \circ b = a \circ b \neq \phi$, so that Condition C yields

$$L(a \circ b) = RLa = La = L(a \circ Lb)\,.$$

Dually,

$$R(a \circ b) = R(Ra \circ b)\,.$$

(A3b) If $La \circ Rb \neq \phi$, then $b \circ a = \phi$ and this yields $Rb \circ La = \phi$.

If $La \circ Rb = \phi$, then by Condition C, $RLa = LRb$, i.e., $La = Rb$ so that $La \circ Rb = Rb \circ La$.

(A3c) If $Ra \circ b = \phi$, then again, $a = \phi$ or $b = \phi$ or $Ra \neq Lb$, whence $b \circ R(a \circ b) = \phi$.

If $Ra \circ b \neq \phi$, then $RRa = Lb$, i.e., $Ra = Lb$. Consequently,

$$Ra \circ b = Lb \circ b = b = b \circ Rb = b \circ R(Lb \circ b) = b \circ R(Ra \circ b) = b \circ R(a \circ b)\,.$$

Thus every categorical semigroup is an f-system; since the converse is trivial, the proof is complete.

Condition C is clearly self-dual. Thus the argument used above to prove A3c dualizes and yields the fact that the identity D3c holds in any categorical semigroup.

Theorem 25. *An f-system is a categorical semigroup if and only if the partial order "$\subseteq$" is trivial, in the sense that for any pair a, b of elements, $a \subseteq b$ if and only if $a = \phi$ or $a = b$.*

Proof. Let a, b be two elements of A such that $\phi \neq a \subseteq b$. Then $b \circ Ra = a \neq \phi$ which, in view of Condition C, yields $Rb = LRa = Ra$. Hence

$$a = b \circ Ra = b \circ Rb = b\,,$$

and the partial order is trivial.

Conversely, suppose that a, b are two elements of an f-system in which the partial order is trivial and that $a \circ b \neq \phi$. Then, first of all we have $R(a \circ b) \neq \phi$, $a \neq \phi$, $b \neq \phi$ and $Lb \neq \phi$. Next, since $\phi \neq R(a \circ b) \subseteq Rb$ and the partial order is the trivial one, we must have $R(a \circ b) = Rb$. By T8, $Lb \subseteq Ra$. Since $Lb \neq \phi$, this yields $Lb = Ra$. Thus the f-system in question satisfies Condition C, i.e., is a categorical semigroup.

We turn now to categories proper. These, although commonly defined in terms of "objects" and "morphisms", can just as readily be defined in terms of morphisms alone [2, p. 37; 4, p. 3; 5, p. 41]; in the latter case, the resulting systems are sometimes called "abstract categories". For our present purpose it is more convenient to have only a single kind of entity, and we will therefore confine our attention to abstract categories. Furthermore, since we are more interested in elucidating the algebraic structure of our constructs than in applying these constructs to "classes" (in the sense of VON NEUMANN-BERNAYS-GÖDEL), we will consider only categories in which the class of morphisms is a set. These are sometimes called "small" categories [5, p. 43]. This restriction (already implicit in the use of the word "set" in various parts of the Introduction and in Definition 1), convenient as it is for avoiding distractions, is in no way essential and could be dropped if so desired. Accordingly we modify the presentations in [4] and [5] (cf. also [2, p. 40]) slightly to make the following:

Definition 9. A *category* is a pair $(A, \circ)$ consisting of a non-empty set A (whose elements will be called *morphisms*), and a partial binary operation "$\circ$" on A, subject to the following conditions:

(1) For any triple of morphisms, a, b, c:

(a) If $a \circ b$ and $b \circ c$ are defined then $(a \circ b) \circ c$ and $a \circ (b \circ c)$ are defined and are equal.

(b) If $a \circ b$ and $(a \circ b) \circ c$ are defined then $b \circ c$ is defined; if $b \circ c$ and $a \circ (b \circ c)$ are defined then $a \circ b$ is defined.

(2) For any morphism a, there exist identity morphisms l_a and r_a such that $l_a \circ a$ and $a \circ r_a$ are defined; here e is an *identity morphism* if $e \circ a = a$ whenever $e \circ a$ is defined and $a \circ e = a$ whenever $a \circ e$ is defined.

It is immediate from (2) that $l_a \circ a = a$ and $a \circ r_a = a$. Moreover it is easily verified that, for any given a, l_a and r_a are unique. Thus we can define unary operations L and R on A via: $La = l_a$ and $Ra = r_a$, for any a in A. Clearly, $La \circ a = a = a \circ Ra$; from the fact that $r_a \circ r_a$ is defined and equal to r_a it follows that $LRa = RRa = Ra$; similarly, $RLa = LLa = La$. Thus A2 is valid in any category. Furthermore, if $a \circ b$ is defined then $(a \circ Ra) \circ b$ is defined; hence $Ra \circ b$ is defined, so that, since Ra is an identity morphism, $Ra = Lb$. Conversely, if $Ra = Lb$ then, since $a \circ Ra$ and $Lb \circ b = Ra \circ b$ are defined, $a \circ b$ is defined. Combining this series of observations yields the following equivalent and rather more perspicuous formulation of Definition 9:

Definition 9′. A *category* is a quadruple $(A, \circ, L, R)$ consisting of a non-empty set A, a partial binary operation "$\circ$" on A, two unary operations L and R on A; and satisfying Condition (1) of Definition 9, Axiom 2, and:

(C′) For all a, b in A, $a \circ b$ is defined if and only if $Ra = Lb$.

Since the operation "$\circ$" is only defined on a (possibly proper) subset of $A \times A$, the pair $(A, \circ)$ of Definition 9′ is a *partial groupoid* [1, p. 1]. There is a standard technique (cf. [1, p. 100]) for converting a partial groupoid $(A, \circ)$ to a (complete) groupoid by adjoining a zero element ϕ to A, and extending $\circ$ to all of $(A \cup \{\phi\}) \times (A \cup \{\phi\})$ by setting $a \circ b = \phi$ whenever $a \circ b$ is not defined within $(A, \circ)$; in particular, $a \circ \phi = \phi \circ a = \phi \circ \phi = \phi$. When this technique is applied to a partial groupoid $(A, \circ)$ of a category, the resulting groupoid $(A \cup \{\phi\}, \circ)$ is a semigroup, i.e., A1 holds [1, Lemma 3.7, p. 100]. Similarly L and R can be extended to all of $A \cup \{\phi\}$ by defining $L\phi = R\phi = \phi$. This process of extending $\circ$, L and R will be called *adjunction of a zero element*. Referring to the above results and comparing Definition 9′ with Definition 8, we immediately have the following:

Theorem 26. *When a zero element* ϕ *is adjoined to a category* $(A, \circ, L, R)$, *the resulting quadruple* $(A \cup \{\phi\}, \circ, L, R)$ *is a categorical semigroup. Conversely, when the zero element* ϕ *of a categorical semigroup* $(A, \circ, L, R)$ *is deleted, the resulting quadruple* $(A - \{\phi\}, \circ, L, R)$ *is a category.*

Combining Theorems 24, 25 and 26 establishes the connection between f-systems and categories: a category is simply an f-system in which the partial order is trivial and from which any zero element has been deleted.

When specialized to categories, Theorem 14 reduces to the representation theorem of EILENBERG-MACLANE [3, Appendix; 4, p. 9], with the single difference that the elements of the category are represented by functions instead of mappings. Since we are dealing with isomorphisms, nothing is lost in this transition from mappings to functions. A simple example may serve to illustrate the point:

Let $\mathscr{X}$ be a category of (sets and) mappings. The elements of $\mathscr{X}$ are thus triples of the form (X, Y, f). For any element $x = (X, Y, f)$ in $\mathscr{X}$, Rx and Lx are the identity mappings,

$$Rx = (X, X, j_X), \quad Lx = (Y, Y, j_Y),$$

where, for any set Z, j_Z is the identity function on Z; and for any pair of elements $x_1 = (X_1, Y_1, f_1)$ and $x_2 = (X_2, Y_2, f_2)$ in $\mathscr{X}$, $x_1 \circ x_2$ is defined if and only if $X_1 = Y_2$, in which case $x_1 \circ x_2 = (X_2, Y_1, f_1 \circ f_2)$. Now let the set A be a proper subset of the set B; let a denote the identity mapping (A, A, j_A) and b the injection mapping (A, B, j_A). Then $a \neq b$, $La = Ra = Rb = a$, and $Lb = (B, B, j_B)$. Next, any c in $\mathscr{X}$ which is such that $Lc \leqq Ra$ is of the form $c = (C, A, g)$, for some set C. It follows that

$$Lc = Ra = a \quad \text{and} \quad a \circ c = Lc \circ c = c,$$

whereas $b \circ c = (C, B, g) \neq c$.

Consequently the functions f_b and f_b representing a and b, respectively, are given by

$$\operatorname{Dom} f_a = \{c \mid Lc = a\}, \quad f_a(c) = c,$$
$$\operatorname{Dom} f_b = \{c \mid Lc = a\}, \quad f_b(c) = b \circ c.$$

Thus $f_a \neq f_b$. In fact, while f_a and f_b have the same domain, the two functions have disjoint ranges and are totally different in behavior: f_a is an identity function, while f_b does not have a single fixed point.

6. Brandt semigroups

The results of the two preceding sections yield a simple characterization of Brandt semigroups, which are related to Brandt groupoids as categorical semigroups are to categories: see [1, pp. 99—101].

Theorem 27. *A Brandt semigroup is a categorical inverse semigroup satisfying the further condition:*

(B) *If a and b are subidentities with $b \neq \phi$, then there exists an element c such that $Rc = a$ and $Lc \subsetneqq b$.*

Proof. Brandt semigroups have been characterized [1, p. 103, Ex. 2] as inverse semigroups in which every non-zero idempotent is primitive [1, p. 76] and which satisfy the additional condition [1, p. 101, Axiom A4]:

(B′) If s and t are non-zero idempotents then there exists an element c such that $s \circ c \circ t \neq \phi$.

In the present context, the primitiveness of non-zero idempotents is equivalent to Condition C, hence to the condition that the semigroup be categorical, while Condition B′ is readily shown to be equivalent to Condition B. This completes the proof.

Condition B is significant in wider contexts. A special case of it was used to obtain one of the representation theorems, Theorem 93 of [11]. Moreover, any f-system which satisfies this condition and contains a non-null constant in the sense of [9; D9] has an identity and is therefore a function semigroup.

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Professor B. SCHWEIZER
Department of Mathematics
University of Massachusetts
Amherst, Mass.

Professor A. SKLAR
Department of Mathematics
Illinois Institute of Technology
Chicago 16, Ill.

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